THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Remark for Tutorial 1

In last tutorial class, I gave the following definition of base:

Definition 1. *A collection of subsets $B \subset \mathfrak{P}(X)$ is called a base if it satisfies the following properties:*

1. $\emptyset, X \in B$;
2. *For any $U, V \in B$ and $x \in U \cap V$, there exists some $W \in B$ such that $x \in W \subset U \cap V$.*

However, this definition is **INCORRECT**. The correct definition should be:

Definition 2. *A collection of subsets $B \subset \mathfrak{P}(X)$ is called a base if it satisfies the following properties:*

1. *For any $x \in X$, there exists some $D \in B$ such that $x \in D$;*
2. *For any $U, V \in B$ and $x \in U \cap V$, there exists some $W \in B$ such that $x \in W \subset U \cap V$.*

For example, for $X = \mathbb{R}$, the set of all open intervals forms a base for X, even through it does not contain the empty set and the whole space. Clearly the conditions in Definition 1 implies the conditions in Definition 2. However it is too strong to have the condition that $\emptyset, X \in B$.

Finally, consider the example $X = \mathbb{R}$ again. The set $B = \{(a, b) \mid a < b\}$ is a base for X and B generates the standard topology on $\mathbb{R}$. Note that in this case, any non-empty intersection of open intervals is again an open interval. Therefore, it is natural to ask:

Question. *Is the following condition equivalent to the condition (2)?*

3. *For any $U, V \in B$ with $U \cap V \neq \emptyset$, we have $U \cap V \in B$.*

Clearly (3) implies (2). Can you construct an example of $B \subset \mathfrak{P}(X)$ that satisfies (1) and (2) but not (3)?